

A Model for Partial Kantian Cooperation*

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Abstract

This work presents a game theoretic model to describe game situations in which there is a partial cooperation among the players. Specifically, we assume that the players partially follow Kant’s “Categorical Imperative”. The model is stated for games with a continuum of players and the basic assumption made is that the participants consider that they belong to virtual groups in which they optimize their actions as if they were bound to follow the same strategy. The relation with the Nash, (Bentham-) Harsanyi, Rawls difference and Roemer solutions is then studied. We derive necessary conditions characterizing the partial Kantian equilibria, using a reduction to a set of optimal control problems. Finally, some examples are given.

1 Introduction

This work studies the behaviour of the players in game situations, in the case where they partially follow Kant’s “categorical imperative” ([1]):

Act only according to that maxim whereby you can, at the same time, will that it should become a universal law.

Several issues may arise. First, the players may have different action sets, their actions may have different impact to the others or they may have different preferences. Hence, a “maxim” is interpreted as a strategy of a player (i.e. a mapping from a state or type to the action set) and not

*This is an ongoing work in early stage. **Any comments or references are very welcome.**

an action¹. Second, in order to take into account that the players have different preferences, the notion of the veil of ignorance within virtual groups will be used ([2] under the name equi-partition and [3]). This notion is probably also implicit in Kant's work. Third, when a veil of ignorance is used the problem of interpersonal comparison of utility arises. However, this is not an issue in a descriptive model, since what is important is how each player perceives the utilities of the others and the players do not need to agree on the scaling of the utilities of the other persons. Finally, the players know that it is not true that all the others will follow their strategy. Hence, it is interesting to study how the players would behave if each one of them assumes that some of the others would follow her strategy.

In [4], [5] the notion of the Kantian Equilibrium is introduced. A set of strategies constitutes a Kantian Equilibrium if no player has a motivation to change her action assuming that the rest of the players would change their actions accordingly (ex. multiplicatively or additively). It turns out that under weak conditions the set of Kantian Equilibria coincides with the Pareto front. However, the Kantian Equilibrium is status-quo dependent (and thus conservative) and the players cannot determine their actions without referring to the actions of the other players. Furthermore, often Pareto front contains fundamentally unjust solutions. It is probably not reasonable to expect from a player that is very much disadvantaged by a solution in the Pareto front to be willing to cooperate with the others while she has the opportunity to improve her position by changing unilaterally her action. [6] and [7] extended [4] in two distinct directions. First, they consider dynamic games and second study game situations with mixed Kantian and Nash players and introduce the notions of (inclusive and exclusive) Kant-Nash equilibria.

In this work the notions of r -Kant-Nash equilibrium and r, h -Kant-Nash equilibrium are introduced and the relation with several known concepts is studied. Then, the r -Kant-Nash equilibria are

¹Let us quote a part of a story in which Woody Allen makes fun of several philosophers.

“No less misguided was Kant, who proposed that we order lunch in such a manner that if everybody ordered the same thing the world would function in a moral way. The problem Kant didnt foresee is that if everyone orders the same dish there will be squabbling in the kitchen over who gets the last branzino. “Order like you are ordering for every human being on earth,” Kant advises, but what if the man next to you doesnt eat guacamole? In the end, of course, there are no moral foods-unless we count soft-boiled eggs.”

From Woody Allen “THUS ATE ZARATHUSTRA” New Yorker JULY 3, 2006

characterized using optimal control problems. To illustrate the application of the results, the fishing game example is used.

All the sets and functions thereafter are assumed to be measurable.

2 The Model

There is a continuum of players, each one of which has an individual type $x_i \in X$ and a social preference type $\theta_i \in \Theta$. The individual type describes both the preferences of a player and the effects of the actions of that player to the costs of the others (position). Denote by D the set of possible individual-social preference pairs, i.e. $D = X \times \Theta$ and by \mathcal{A} a sigma algebra on D . Let us denote the distribution of the position-preference pairs by p .

The cost function of each player is given by:

$$J_i = J(u_i, \bar{u}, x_i), \quad (1)$$

where \bar{u} is statistic of the players' actions:

$$\bar{u} = \int_D g(u(x, \theta), x) dp. \quad (2)$$

Let us then describe the idea of a virtual group.

- (i) Each player i assumes that she is associated with a virtual group of players. This group is described by a sub-probability measure $r(\cdot, x_i, \theta_i)$ on (D, \mathcal{A}) . For every $A \in \mathcal{A}$, the sub-probability measure should satisfy $r(A, x_i, \theta_i) \leq p(A)$. If $r(D, x_i, \theta_i) \neq 0$, let us denote by $\bar{r}(\cdot, x_i, \theta_i)$ the probability measure $r(\cdot, x_i, \theta_i)/r(D, x_i, \theta_i)$. If $r(D, x_i, \theta_i) = 0$ then we define $\bar{r}(\cdot, x_i, \theta_i) = \delta_{x_i, \theta_i}$, where δ_{x_i, θ_i} is the Dirac measure.
- (ii) Player i assumes that all the members of the virtual group are bound to use the same strategy $u = \gamma_{x_i, \theta_i}(x) = \gamma(x, x_i, \theta_i)$. That is, Player i assumes that all the players in her group having individual type x will have an action $\gamma(x, x_i, \theta_i)$.
- (iii) The aim of the virtual group is to minimize:

$$\tilde{J}_i(\gamma, \bar{\gamma}) = \frac{1}{\beta_{\theta_i}} \ln E \{ \exp[\beta_{\theta_i} J(\gamma(x', \theta_i), \bar{u}_i, x') w_{\theta_i}(x, x')] \}, \quad (3)$$

where x' is a random variable following the X marginal of $\bar{r}(\cdot, x_i, \theta_i)$, the factor w_{θ_i} is a weighting function indicating the relative importance of the several positions in the group and

$\beta_{\theta_i} \in [-\infty, \infty]$ is a risk factor. The value of \bar{u} corresponds to the g -mean value of the actions of all the players assuming that the members of the group are using $u = \gamma(x, x_i, \theta_i)$ and the strategy of the players not belonging to the group is given by $\bar{\gamma}(x, \theta)$. Thus,

$$\bar{u} = T_{x_i, \theta_i}(\gamma, \bar{\gamma}) = \int_D g(\bar{\gamma}(x, \theta), x)(p(d(x, \theta)) - r(d(x, \theta), x_i, \theta_i)) + \int_D g(\gamma(x, x_i, \theta_i), x)r(dx, x_i, \theta_i) \quad (4)$$

Remark 1 (i) *The virtual groups defined have some points in common with the virtual co-movers model of [6]. Specifically in the virtual co-movers model each player assumes that if she changes her strategy, a subset of the others would also change theirs accordingly.*

(ii) *The definition of the members of the virtual group of each player offers a lot of flexibility. The two extreme cases are the case where $r = 0$ and the group of each player consists only of herself and the case where $r(\cdot, x, \theta) = p(\cdot)$. In the intermediate cases the quantity $r(\cdot, x, \theta)$ may relate to race, class, religion, gender, ethnicity, ideology, nationality, sexual orientation, culture or language².*

(iii) *The virtual groups, the way they defined are purely imaginary. Thus, the fact that a player i assumes that an other player j is included in her virtual group does not necessarily imply that the virtual group of player j includes i .*

(iv) *It is probably useful to discriminate between the individual position and individual preferences, both contained in x . In this case it is possible that within a virtual group the players are allowed to use strategies depending only on their individual position and not their individual preferences. In this case strategies of the form $u = \gamma(y, x_i, \theta_i)$, where $y = h(x, x_i, \theta_i)$ will be considered.*

The players do not necessarily agree on which part of the x variables of the other players correspond to individual positions and which to individual preferences. Let us note that often people tend to overestimate the effect of the character (preferences) of the other persons and underestimate situational factors (position), a phenomenon known as the fundamental attribution error [8]. However, the virtual group of each player belongs exclusively in her perception (or imagination) and thus the function h is defined to depend on x_i, θ_i .

² From the identity politics article of Wikipedia

Definition 1 A set of strategies $u = \bar{\gamma}(x, \theta)$ is an r -Kant-Nash equilibrium if for each $(x_i, \theta_i) \in D$ a solution $\gamma(x, x_i, \theta_i)$ of the optimization problem:

$$\underset{\gamma(\cdot, x_i, \theta_i)}{\text{minimize}} \quad \tilde{J}_i(\gamma, \bar{\gamma}) \quad (5)$$

satisfies $\gamma(x_i, x_i, \theta_i) = \bar{\gamma}(x_i, \theta_i)$.

A possibly useful alternative is to define the r -Kant-Nash equilibrium assuming that the optimization problems in the virtual groups are solved within the class of strategies of the form $u = \gamma(y, x_i, \theta_i)$.

Definition 2 A set of strategies $u = \bar{\gamma}(x, \theta)$ is an h, r -Kant-Nash equilibrium if for each $(x_i, \theta_i) \in D$ a solution $\gamma(x, x_i, \theta_i)$ of :

$$\begin{aligned} &\underset{\gamma(\cdot, x_i, \theta_i)}{\text{minimize}} \quad \frac{1}{\beta_{\theta_i}} \ln E \{ \exp[\beta_{\theta_i} J(\gamma(h(x', x_i, \theta_i), \theta_i), \bar{u}_i, h(x', x_i, \theta_i)) w_{\theta_i}(x, x')]] \} , \\ &\text{subj. to} \quad \bar{u} = T_{x_i, \theta_i}(\gamma(h(\cdot, x_i, \theta_i), \bar{\gamma})) \end{aligned} \quad (6)$$

within the class of strategies of the form $u = \gamma(y, x_i, \theta_i)$ satisfies $\gamma(x_i, x_i, \theta_i) = \bar{\gamma}(x_i, \theta_i)$.

Remark 2 Let us comment on the role of the several components of the definitions.

- (i) The weighting factor w_{x_i, θ_i} may have two discrete roles. At first, Player i may believe that in her virtual group some subgroup of players should be favored over the others. A second and probably more important role is that the **interpersonal comparison of utility** problem may be resolved using a weighing factor.
- (ii) The function h has also two possibly distinct roles. At first, player i probably cannot understand or does not know the preference part of the states of the other players belonging to her virtual group. Secondly, it is possible that Player i despite the fact that feels that she belongs to a virtual group involving another player j , she does not want to optimize for a certain preference of Player j that she believes as not good or important for the group.

3 Special Cases

The notion of r -Kant-Nash equilibrium has several interesting special cases. In the first four cases Θ is a singleton.

- (i) The mean field **Nash equilibrium**. Assuming that $r \equiv 0$ and $\beta_\theta = 0$ each player uses her best response to the actions of the other players. Hence, for these values the r -Kant-Nash equilibrium coincides with the mean field Nash equilibrium.
- (ii) The **(Bentham-) Harsanyi** solution. Assume that $\beta_\theta = 0$ and $r(\cdot, x, \theta) = p(\cdot)$. Then, each player is risk neutral and optimizes for the mean cost (or equivalently the sum of the costs) of all the players. This solution coincides with the solution proposed in [2].
- (iii) The **Rawls** solution. Assume that $\beta_\theta = \infty$ and $r(\cdot, x, \theta) = p(\cdot)$. In this case, all the players minimize the cost function of the worse of participant, i.e. they use the minimax rule. This solution coincides with the Rawls difference solution [3]
- (iv) Assume that $\beta_\theta = -\infty$ and $r(\cdot, x, \theta) = p(\cdot)$. In this case, all the players minimize the cost function of the **better off participant**. This optimization procedure makes sense when the better off participant represents the others (for example the best athlete).
- (v) Efficient cooperation within **coalitions**. Consider the coalitions $C_1, \dots, C_N \subset D$ and assume that C_j are a partition of D . Further assume that the virtual groups are the same with the coalitions. That is, for $(x_i, \theta_i) \in C_j$ it holds $r(A, x_i, \theta_i) = p(A \cap C_j)/p(C_j)$. Finally assume that $w_{\theta_i}(x, x') = g_{\theta_i}(x')$. Then, within each coalition the players jointly optimize for a weighted sum of their costs, and thus within each coalition there is an efficient cooperation.
- (iv) **Simple Kantian optimization**. The notion of simple Kantian optimization, defined in [5], studies cases where each player optimizes for her own cost assuming that all the other players would have exactly the same action with her. This is a limit case of the h, r -Kant-Nash equilibrium assuming that $r(\cdot, x, \theta) = p(\cdot)$, the range of the function h is a singleton and $w(x, x') \rightarrow \delta_x(x')$ where δ stands for the Dirac function.
- (v) The relation with the altruistic (other regarding) behaviour is illustrated in the following example.

Example 1 (The Fishing Game) *There is a large number N of fishers each each of which has a cost function:*

$$J_i = u_i^2 - \left(1 - \frac{1}{N-1} \sum_{j \neq i} u_j\right) u_i, \quad (7)$$

where the first term corresponds to the effort of the fisher i and the second on the revenues.

The altruistic (other regarding) cost for player i is:

$$\bar{J}_i = (1 - \alpha/2)J_i + (\alpha/2) \sum_{j \neq i} J_j = (1 - \alpha/2)(u_i^2 - u_i) + \frac{1}{N-1}u_i \sum_{j \neq i} u_j + f(u_{-i}), \quad (8)$$

with $\alpha \in [0, 1]$.

The Nash equilibrium of the altruistic game is given by:

$$u_i = \frac{2 - \alpha}{6 - 2\alpha} \quad (9)$$

Let us then consider the partial Kantian strategy for the game with a continuum of players. Assume that $\Theta = \{\theta\}$ and $X = \{x\}$ are singletons. Further assume that each player considers as her virtual group a fraction α of the other players. Then, the r -Kant-Nash equilibrium in the form $u = \gamma$ is characterized by:

$$\frac{\partial}{\partial \gamma} J(u, \bar{u}) = \frac{\partial}{\partial u} J(u, \bar{u}) + \frac{\partial}{\partial \bar{u}} J(u, \bar{u}) \frac{\partial \bar{u}}{\partial \gamma} = 0, \quad (10)$$

Hence,

$$2u - 1 + \bar{u} + (1 - \alpha)u = 0. \quad (11)$$

Due to symmetry:

$$u = \frac{1}{3 + \alpha}. \quad (12)$$

Figure 1 compares the actions and the costs of the players in the cases of an altruistic versus a partially Kantian behaviour. It turns out that the Kantian cooperation is more effective than the altruism. \square

(vi) The relation with **Roemer's Kantian equilibrium** is illustrated in the following example:

Example 2 (The Fishing Game: Continued) Consider again the Fishing Game example. Assume also that in a previous iteration of the game (status quo) the players used actions u_i^{prev} . Denote by x_i the state of each player to be the action that she had in the status quo.

Let the actions of the game to be denoted by v_i and relate to the original Fishing game with $u_i = x_i v_i$. Then, assuming $\beta_\theta = 0$, $r(\cdot, x, \theta) = p(\cdot)$ and $h(x, x_i, \theta_i) = x_i$ an h, r -Kant-Nash equilibrium of the game is a Roemer's Kantian equilibrium. \square

This analogy illustrates that the Kantian equilibrium is status quo dependent. Furthermore, the choice of the multiplicative representation of the game is chosen such that the outcome

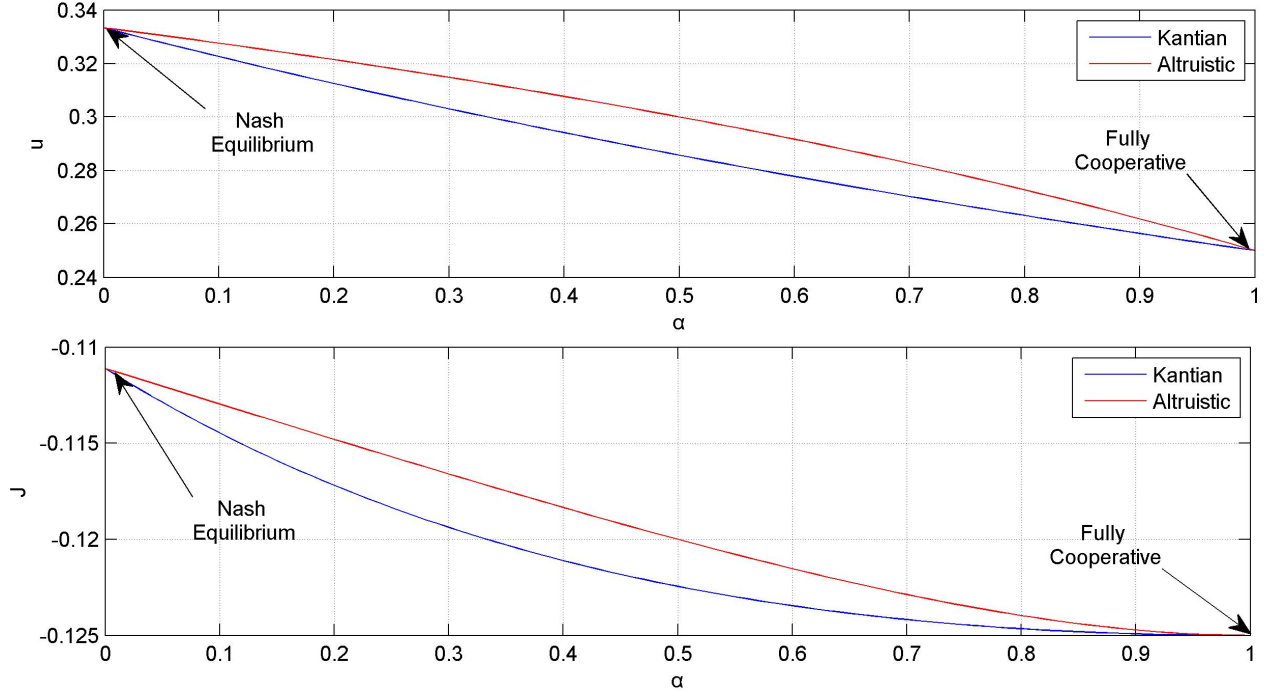


Figure 1: Comparison of the actions and the Costs of the players when they use a partially Kantian vs an Altruistic criterion

belongs to the Pareto front. Hence, Kantian equilibrium depends on the “experience” in two distinctive ways, i.e. the action changing rule and the status quo. Kant in his Groundwork [1] strongly opposes to the use of “experience” in order to derive normative rules. However, this fact in no case implies that people are not using their experience to determine how to behave. What it probably means is that in Roemer’s work the actions of the players depend too much on the experience while in this work too little.

4 Reformulation as Optimal Control Problems

In this section we characterize r -Kant-Nash equilibria under the assumption that Θ is a singleton, $X = [0, T_f]$, $\beta = 0$, and that all the measures are absolutely continuous with respect to the Lebesgue measure.

Assuming that the players that do not belong to i ’s virtual group follow a strategy $u_j = \bar{\gamma}(x_j)$, the optimization problem (3) is written as:

$$\underset{\gamma}{\text{minimize}} \left\{ \int_0^{T_f} J \left(\gamma(x'), \bar{u}^{-x_i} + \int_0^{T_f} g(\gamma(z), z) r(z, x_i) dz, x' \right) w(x_i, x') r(x', x_i) dx' \right\}, \quad (13)$$

where,

$$\bar{u}^{-x_i} = \int_0^{T_f} g(\bar{\gamma}(z), z) (p(z) - r(z, x_i)) dz, \quad (14)$$

and (with a slight abuse of notation) r denotes the density of the measure $r(\cdot, x_i)$ with respect to the Lebesgue measure.

The optimization problem (13), assuming \bar{u}^{-x_i} as given, can be reformulated as an optimal control problem using the state x' as a virtual time.

Proposition 1 *The optimization problem (13) is equivalent to the optimal control problem:*

$$\begin{aligned} & \underset{u^{x_i}(t)}{\text{minimize}} \quad \int_0^{T_f} L^{x_i}(u^{x_i}, \bar{u}^{-x_i} + \chi_2^{x_i}, t) dt \\ & \text{subject to} \quad \dot{\chi}_1^{x_i} = g(u^{x_i}, t) r(t, x_i), \quad \chi_1(0) = 0 \\ & \quad \quad \quad \dot{\chi}_2^{x_i} = 0, \quad \chi_2^{x_i}(0) : \text{free} \\ & \quad \quad \quad \chi_1^{x_i}(T_f) = \chi_2^{x_i}(T_f), \end{aligned} \quad (15)$$

where

$$L^{x_i}(u, v, t) = J(u, v, t) w(x_i, t) r(t, x_i)$$

Proof: Observe that $\int_0^{T_f} g(u^{x_i}, t) r(t, x_i) dt = \chi_1^{x_i}(T_f) = \chi_2(T_f) = \chi_2(t)$. \square

Necessary conditions can be derived using Pontryagin minimum principle. It turns out that the problem has a special structure and the optimal control law is characterized by a pair of algebraic equations instead of a two point boundary value problem. The Hamiltonian is given by:

$$H^{x_i} = L^{x_i}(u^{x_i}, \bar{u}^{-x_i} + \chi_2^{x_i}, t) + p_1^{x_i} g(u^{x_i}, t) r(t, x_i). \quad (16)$$

The costate equations are given by:

$$\dot{p}_1^{x_i} = 0, \quad \dot{p}_2^{x_i} = -\frac{\partial L^{x_i}}{\partial v}(u^{x_i}, \bar{u}^{-x_i} + \chi_2^{x_i}, t) \quad (17)$$

(where v is the second argument of L^{x_i}) and the boundary conditions by:

$$p_2^{x_i}(0) = 0, \quad p_1^{x_i}(T_f) + p_2^{x_i}(T_f) = 0. \quad (18)$$

Let us assume that there is a unique minimizer $u^{x_i} = l(t, \chi_2, p_1, \bar{u}^{-x_i}, x_i)$ of H^{x_i} with respect to u^{x_i} . In order to characterize the optimal controller it remains to determine the constants $p_1^{x_i}$ and $\chi_2^{x_i}$. A pair of algebraic equations will be derived.

Combining $\dot{p}_1^{x_i} = 0$, $p_2^{x_i}(0) = 0$ and $p_1^{x_i}(T_f) = -p_2^{x_i}(T_f)$ we get:

$$p_1^{x_i} = -p_2^{x_i}(T_f) = \int_0^{T_f} \frac{\partial L^{x_i}(l(t, \chi_2^{x_i}, p_1^{x_i}, \bar{u}^{-x_i}, x_i), \chi_2^{x_i} + \bar{u}^{-x_i}, t)}{\partial v} dt \quad (19)$$

The right hand side of (19) is a known function of $\chi_2^{x_i}$, $p_1^{x_i}$ and \bar{u}^{-x_i} .

The second algebraic equation is obtained combining $\dot{\chi}_2^{x_i} = 0$, $\chi_1^{x_i}(0) = 0$ and $\chi_1^{x_i}(T_f) = \chi_2^{x_i}(T_f)$:

$$\chi_2^{x_i} = \int_0^{T_f} g(l(t, \chi_2^{x_i}, p_1^{x_i}, \bar{u}^{-x_i}, t, x_i) r(t, x_i)) dt, \quad (20)$$

where the right hand side of (20) is again a known function of $\chi_2^{x_i}$, $p_1^{x_i}$ and \bar{u}^{-x_i} .

Proposition 2 *Assume that $\bar{\gamma}(x)$ is an r -Kant-Nash equilibrium. Further assume that there is a unique minimizer l of H^{x_i} for any $x_i \in X$. Then, there exist functions $\chi_2 : X \rightarrow \mathbb{R}$, $p_1 : X \rightarrow \mathbb{R}$ and $\bar{u}^- : X \rightarrow \mathbb{R}$ satisfying, (19), (20) and:*

$$\bar{u}^{-x_i} = \int_0^{T_f} g((l(t, \chi_2^t, p_1^t, \bar{u}^{-t}, x_i), t)(p(t) - r(t, x_i))) dt, \quad (21)$$

such that $\bar{\gamma}(x_i) = l(x_i, \chi_2, p_1, \bar{u}^{-x_i}, x_i)$ for any $x_i \in X$.

Proof: Immediate. □

Thus, an r -Kant-Nash equilibrium is characterized by a couple of algebraic equations and an integral equation.

4.1 Equilibrium in a Quadratic Game

Let us consider again the Fishing Game example assuming players with different efficiencies (for example a fisher is more experienced than another or he has a better boat). We assume that Θ is a singleton, $X = [0, 1]$ and the players have a uniform distribution. The cost function for each player is given by:

$$J_i = u_i^2 - (1 - \bar{u})\xi(x_i)u_i, \quad (22)$$

where:

$$\bar{u} = \int_0^1 u(x)\xi(x)dx \quad (23)$$

and $\xi(x) > 0$ is the efficiency of a player with state x .

We shall compute the r -Kant-Nash equilibrium assuming that $r(x', x) = 0$ implies $w(x', x) = 0$, that is if a player with state x considers a player another player with state x' to belong to his virtual group, he does not assign him a zero weight.

In this example, the optimal control problems are LQ and thus the minimum principle necessary conditions are also sufficient. The Hamiltonian is given by:

$$H^{x_i} = [u^2 - (1 - \bar{u}^{-x_i} - \chi_2^{x_i})\xi(t)u] w(t, x_i)r(t, x_i) + p_1^{x_i}\xi(t)r(t, x_i)u \quad (24)$$

The optimal control u is given by:

$$u = l(t, \chi_2^{x_i}, p_1^{x_i}, u^{-x_i}) = \frac{1}{2}(1 - \bar{u}^{-x_i} - \chi_2^{x_i} - p_1^{x_i}/w(t, x_i))\xi(t) \quad (25)$$

Equation (20) is written as:

$$\chi_2^{x_i} = \frac{1}{2} \int_0^1 (1 - \bar{u}^{-x_i} - \chi_2^{x_i} - p_1^{x_i}/w(t, x_i))\xi^2(t)r(t, x_i)dt \quad (26)$$

or:

$$\chi_2^{x_i} = \frac{(1 - \bar{u}^{-x_i})C_1^{x_i} - p_1C_2^{x_i}}{2 + C_1^{x_i}} \quad (27)$$

where:

$$C_1^{x_i} = \int_0^1 \xi^2(t)r(t, x_i)dt \quad \text{and} \quad C_2^{x_i} = \int_0^1 \xi^2(t)r(t, x_i)/w(t, x_i)dt \quad (28)$$

Equation (19) is written as:

$$p_1^{x_i} = \frac{1}{2} \int_0^1 ((1 - \bar{u}^{-x_i} - \chi_2^{x_i})w(t, x_i) - p_1)\xi^2(t)r(t, x_i)dt. \quad (29)$$

Equivalently:

$$2p_1^{x_i} = (1 - \bar{u}^{-x_i})C_3^{x_i} - \chi_2^{x_i}C_3^{x_i} - p_1^{x_i}C_1^{x_i}, \quad (30)$$

where:

$$C_3^{x_i} = \int_0^1 \xi^2(t)r(t, x_i)w(t, x_i)dt. \quad (31)$$

Solving (27), (30) for $\chi_2^{x_i}$, $p_1^{x_i}$ we obtain:

$$\chi_2^{x_i} = \frac{(C_1^{x_i})^2 + 2C_1^{x_i} - C_2^{x_i}C_3^{x_i}}{(C_1^{x_i})^2 + 2C_1^{x_i} - C_2^{x_i}C_3^{x_i} + 4}(1 - \bar{u}^{-x_i}), \quad (32)$$

$$p_1^{x_i} = \frac{2C_3^{x_i}}{(C_1^{x_i})^2 + 2C_1^{x_i} - C_2^{x_i}C_3^{x_i} + 4}(1 - \bar{u}^{-x_i}). \quad (33)$$

In what follows, in order to simplify the computations we assume that $w(x, x') = 1$. Under this assumption, it holds $C_1^{x_i} = C_2^{x_i} = C_3^{x_i} = C(x_i)$ and:

$$\chi_2^{x_i} = p_1^{x_i} = \frac{C(x_i)}{2C(x_i) + 2}(1 - \bar{u}^{-x_i}). \quad (34)$$

Furthermore,

$$u^{x_i}(t) = \frac{1}{2}(1 - \bar{u}^{-x_i}) \frac{\xi(t)}{C(x_i) + 1}. \quad (35)$$

Equation (21) becomes:

$$\bar{u}^{-x_i} = \int_0^1 \frac{1}{2}(1 - \bar{u}^{-t}) \frac{\xi^2(t)}{C(t) + 1} (1 - r(t, x_i)) dt, \quad (36)$$

which is a linear Fredholm integral equation of second kind.

Example 3 *In this example we assume that $r(x, x') = \alpha$ (a uniform (sub)-distribution). Equation (36) implies that \bar{u}^{-x_i} is independent of x_i . Thus, denoting by \bar{u}^- this constant we obtain:*

$$\bar{u}^- = (1 - \bar{u}^-) \frac{1 - \alpha}{2} \int_0^1 \frac{\xi^2(t)}{C(t) + 1} dt. \quad (37)$$

Thus,

$$u^{x_i}(x_i) = \frac{1}{2 + (1 - \alpha) \int_0^1 \frac{\xi^2(t)}{C(t) + 1} dt} \frac{\xi(x_i)}{C(x_i) + 1}. \quad (38)$$

Hence, the actions of the players scale down uniformly as α increases. □

Example 4 *In this example we assume that:*

$$r(x, x') = \begin{cases} \alpha & \text{if } |x - x'| \leq 0.3 \text{ and } x \leq 0.9 \\ 0 & \text{otherwise} \end{cases} \quad (39)$$

The solution of the integral equation (36) can be approximated using a linear system with a high order. The actions of the players, as well as the cost for the participants of the game are illustrated in figures 2,3.

□

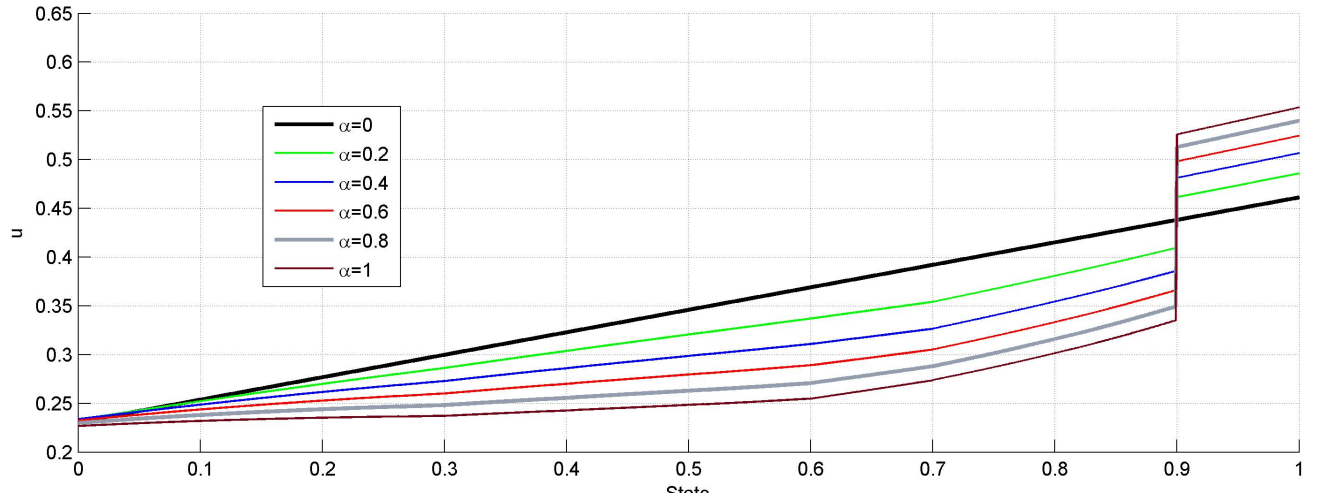


Figure 2: The actions of the several players for different values of α

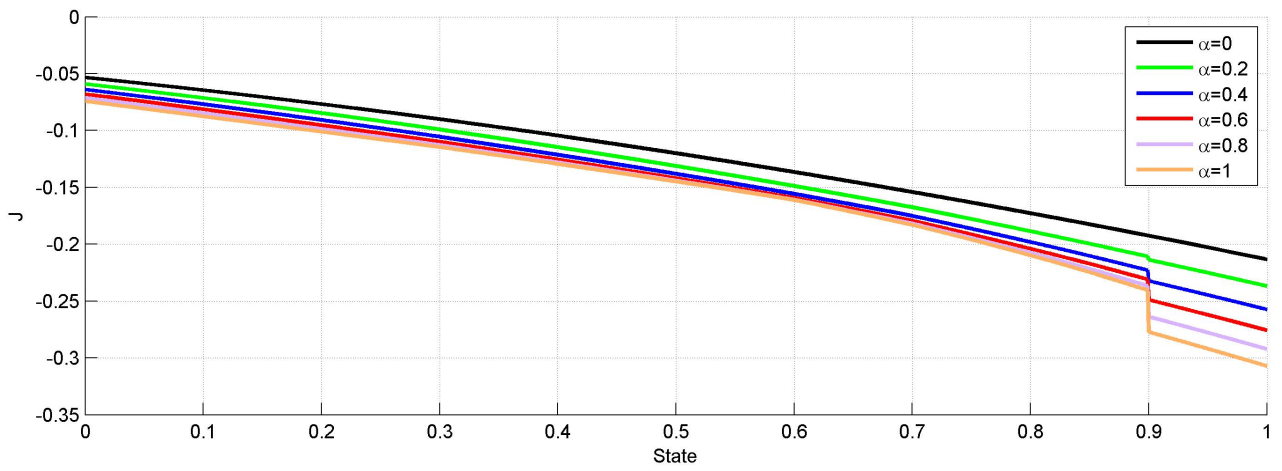


Figure 3: The cost for the several players for different values of α

5 An example of an h, r -Kant-Nash equilibrium

We consider again the Fishing Game example involving players with different efficiencies who value differently their time. Assume that $X = \{1, 2\} \times \{1, 2\}$ and that:

$$J_i = x_i^2 u_i^2 - (1 - \bar{u}) x_i^1(x_i) u_i, \quad (40)$$

where x_i^1 is the efficiency of Player i and x_i^2 determines how much Player i values her working time. Further, assume that $p(1, 1) = p_1 = 0.1$, $p(1, 2) = p_2 = 0.2$, $p(2, 1) = p_3 = 0.3$ and $p(2, 2) = p_4 = 0.4$. In this section, we shall compare the r -Kant-Nash equilibrium with the h, r -Kant-Nash equilibrium, assuming that there is only a single type of social preferences θ and that $r(\cdot, x, \theta)$ is a uniform sub-probability measure with total mass α .

Example 5 (The r -Kant-Nash equilibrium) *In order to derive the equations characterizing a set of strategies $\gamma(1, 1) = u_1^{KN}$, $\gamma(1, 2) = u_2^{KN}$, $\gamma(2, 1) = u_3^{KN}$, $\gamma(2, 2) = u_4^{KN}$ constituting an r -Kant-Nash equilibrium, consider the cost of the virtual group of player i :*

$$\begin{aligned} \tilde{J}_i = & p_1 u_1^2 + 2p_2 u_2^2 + p_3 u_3^2 + 2p_4 u_4^2 - [1 - \alpha(p_1 u_1 + p_2 u_2 + 2p_3 u_3 + 2p_4 u_4) - \\ & - (1 - \alpha)(p_1 u_1^{KN} + p_2 u_2^{KN} + 2p_3 u_3^{KN} + 2p_4 u_4^{KN})] (p_1 u_1 + p_2 u_2 + 2p_3 u_3 + 2p_4 u_4), \end{aligned} \quad (41)$$

or in compact form:

$$\tilde{J}_i = u^T (S + \alpha l l^T) u - (1 - (1 - \alpha) l^T u^{KN}) l^T u, \quad (42)$$

where $S = \text{diag}(p_1, 2p_2, p_3, 2p_4)$, $l = [p_1, p_2, 2p_3, 2p_4]^T$, $u^{KN} = [u_1^{KN}, u_2^{KN}, u_3^{KN}, u_4^{KN}]^T$ and $u = [u_1, u_2, u_3, u_4]^T$. A set of strategies u^{KN} is an r -Kant-Nash equilibrium if the u^{KN} minimizes (42) with respect to u . Equivalently, u^{KN} is an r -Kant-Nash equilibrium if:

$$(S + \alpha l l^T) u^{KN} = \frac{1}{2} (1 - (1 - \alpha) l^T u^{KN}) l \quad (43)$$

After straightforward calculations we get:

$$u^{KN} = (2S + (1 + \alpha) l l^T)^{-1} l \quad (44)$$

The actions of the players and the costs are illustrated in Figures 4 and 5. □

Example 6 (The h, r -Kant-Nash equilibrium) *Consider then the function $h : X \times X \rightarrow X$ with $h((x'^1, x'^2), (x^1, x^2)) = (x'^1, x^2)$. That is, each player acts as is a potion α of the other players*

would follow her strategy taking into account the effectiveness of the actions of the others but not how they value their time³.

Let us then characterize an h, r -Kant Nash equilibrium u^{KN} . There are two kinds of virtual groups. The virtual group of a player with $x_i^2 = 1$ has cost:

$$\begin{aligned} \tilde{J}_i = & p_1(u_1^{t1})^2 + p_2(u_2^{t1})^2 + p_3(u_3^{t1})^2 + p_4(u_4^{t1})^2 - [1 - \alpha(p_1u_1^{t1} + p_2u_2^{t1} + 2p_3u_3^{t1} + 2p_4u_4^{t1}) - \\ & - (1 - \alpha)(p_1u_1^{KN} + p_2u_2^{KN} + 2p_3u_3^{KN} + 2p_4u_4^{KN})](p_1u_1^{t1} + p_2u_2^{t1} + 2p_3u_3^{t1} + 2p_4u_4^{t1}), \end{aligned} \quad (45)$$

where $u_1^{t1} - u_4^{t1}$ are the actions $\gamma((1, 1), x_i), \gamma((1, 2), x_i), \gamma((2, 1), x_i), \gamma((2, 2), x_i)$ in the virtual group of a player having $x_i^2 = 1$. In compact form:

$$\tilde{J}_i = (u^{t1})^T (S_1 + \alpha ll^T) u^{t1} - (1 - (1 - \alpha)l^T u^{KN}) l^T u^{t1}, \quad (46)$$

where $S_1 = \text{diag}(p_1, p_2, p_3, p_4)$, $l = [p_1, p_2, 2p_3, 2p_4]^T$, $u^{KN} = [u_1^{KN}, u_2^{KN}, u_3^{KN}, u_4^{KN}]^T$ and $u^{t1} = [u_1^{t1}, u_2^{t1}, u_3^{t1}, u_4^{t1}]^T$. Similarly, the virtual group of a player with $x_i^2 = 2$ has cost:

$$\tilde{J}_i = (u^{t2})^T (S_2 + \alpha ll^T) u^{t2} - (1 - (1 - \alpha)l^T u^{KN}) l^T u^{t2}, \quad (47)$$

where $S_2 = 2\text{diag}(p_1, p_2, p_3, p_4)$ and $u^{t2} = [u_1^{t2}, u_2^{t2}, u_3^{t2}, u_4^{t2}]^T$.

The values for u^{t1} and u^{t2} which minimize (46) and (47) respectively satisfy the following equations:

$$\begin{aligned} 2(S_1 + \alpha ll^T) u^{t1} + (1 - \alpha) l l^T u^{KN} &= l, \\ 2(S_2 + \alpha ll^T) u^{t2} + (1 - \alpha) l l^T u^{KN} &= l, \end{aligned}$$

Coupled with the consistency conditions $u_1^{KN} = u_1^{t1}$, $u_3^{KN} = u_3^{t1}$ and $u_2^{KN} = u_2^{t2}$, $u_4^{KN} = u_4^{t2}$, the h, r -Kant-Nash equilibrium is characterized by:

$$\begin{bmatrix} 2(S_1 + \alpha ll^T) & 0 & (1 - \alpha) ll^T \\ 0 & 2(S_2 + \alpha ll^T) & (1 - \alpha) ll^T \\ \text{diag}(1, 0, 1, 0) & \text{diag}(0, 1, 0, 1) & -I_4 \end{bmatrix} \begin{bmatrix} u^{t1} \\ u^{t2} \\ u^{KN} \end{bmatrix} = \begin{bmatrix} l \\ l \\ 0 \end{bmatrix} \quad (48)$$

The actions of the players and the costs are illustrated in Figures 4 and 5. □

³ For example a player of type $(2, 1)$, i.e. an efficient player how has a small value for her working time, may accept not to overfish because she doesn't want also the others to overfish taking into account the efficiency of the others. However, she may not be wiling to take into account that some of the other fishers value more their working time. That is, she may not be wiling to reduce her effort because some of the others are "lazy".

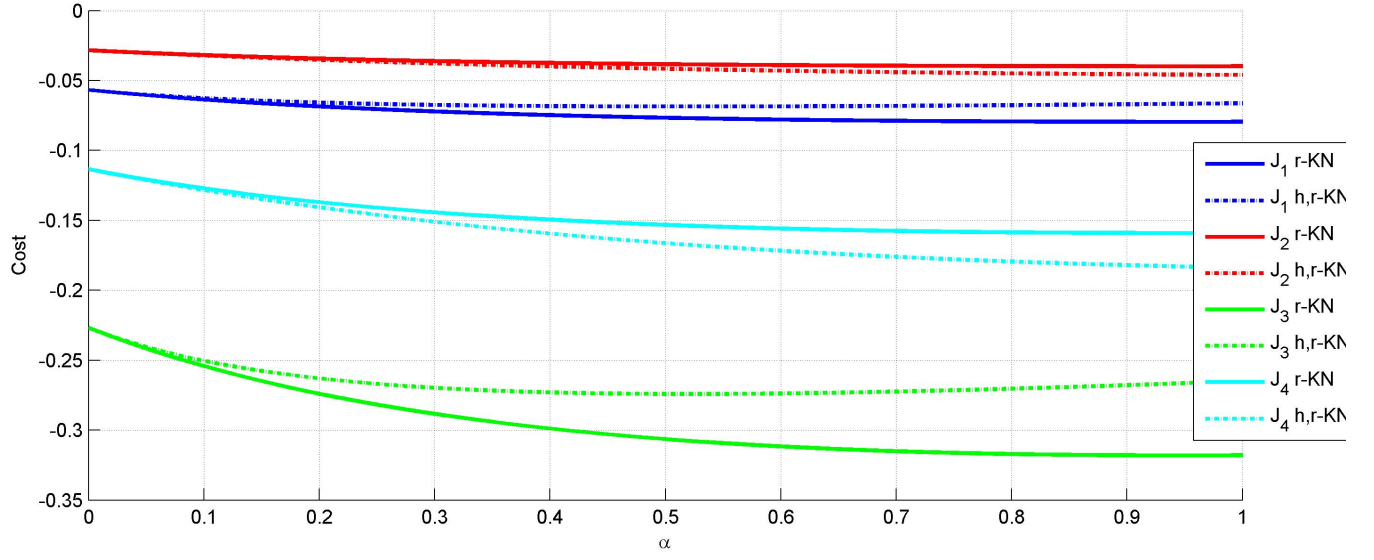


Figure 4: The cost of the players of several types for the cases of r -Kant-Nash and h, r -Kant-Nash equilibria

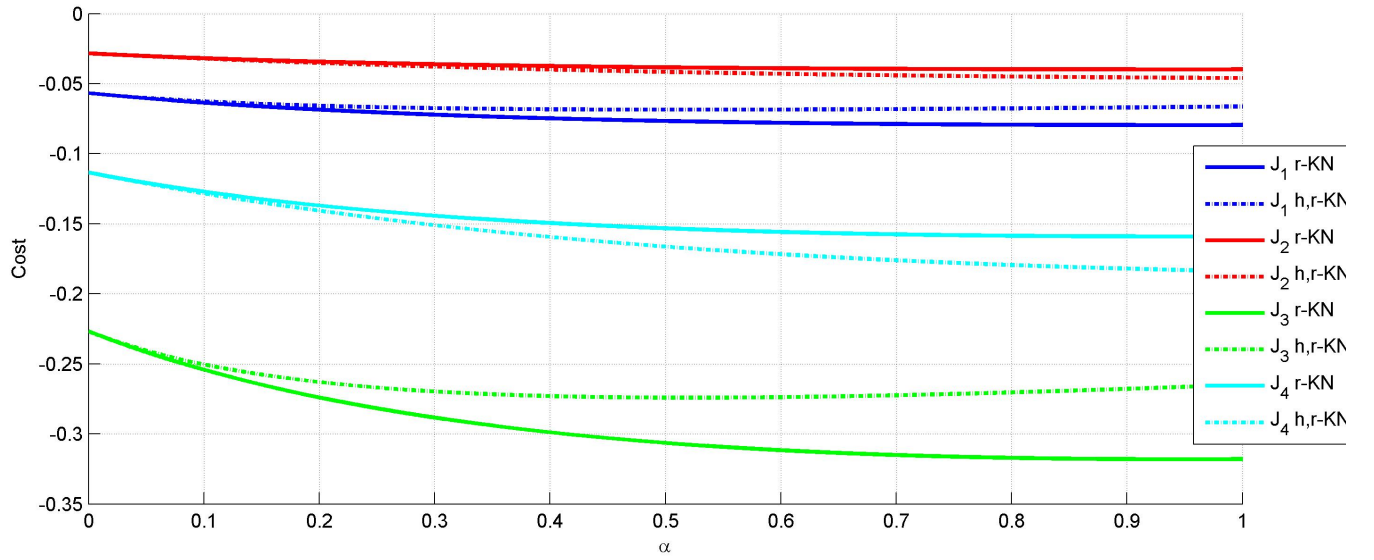


Figure 5: The actions of the players of several types for the cases of r -Kant-Nash and h, r -Kant-Nash equilibria

6 Conclusion and Future Directions

The notions of r -Kant-Nash equilibrium and h, r -Kant-Nash equilibrium were introduced and compared with other notions. Necessary conditions, based on a reduction to a set of optimal control problems, can be derived for cases of games where the possible states admit an 1-dimensional representation. Some examples of quadratic games with a finite number of types was analyzed and r -Kant-Nash and h, r -Kant-Nash equilibria were computed using systems of linear equations.

A possible extension of this work is to study games with a finite number of players. In this case we may assume that the virtual group of each player is stochastic and that each player determines her action before she learns the realization of her virtual group. Another direction for future research is to extend the current model to Dynamic Games.

References

- [1] I. Kant, “Groundwork for the metaphysics of morals,” 1785.
- [2] J. C. Harsanyi, *Cardinal welfare, individualistic ethics, and interpersonal comparisons of utility*. Springer, 1980.
- [3] J. Rawls, *Justice as fairness: A restatement*. Harvard University Press, 2001.
- [4] J. E. Roemer, “Kantian optimization: A microfoundation for cooperation,” *Journal of Public Economics*, vol. 127, pp. 45–57, 2015.
- [5] —, “How we (do and could) cooperate.”
- [6] A. Ghosh and N. Van Long, “Kants rule of behavior and kant-nash equilibria in games of contributions to public goods,” 2015.
- [7] N. Van Long, “Kant-nash equilibrium in a dynamic game of climate change mitigations,” 2015.
- [8] E. Aronson, T. D. Wilson, and R. M. Akert, *Social Psychology*. Pearson Education, 2013.